Math 250

## 5.5 Integration By Substitution - 1st

Objectives

1) Use a change of variables (substitution) to find an indefinite integral

- Sometimes this can be done by pattern recognition
- A specific and frequently-occurring pattern is called the Generalized Power Rule

Differentiate (using Chain Rule):

① $F(x) = \sqrt{x^3-2}$

$F'(x) = \frac{1}{2}(x^3-2)^{-1/2}(3x^2)$

$\boxed{F'(x) = \dfrac{3x^2}{2\sqrt{x^3-2}}}$ or $\boxed{F'(x) = \frac{3}{2}x^2(x^3-2)^{-1/2}}$

This means:

$$\int \frac{3x^2}{2\sqrt{x^3-2}}\,dx = \sqrt{x^3-2} + C$$

and $\int \frac{3}{2}x(x^3-2)^{-1/2}\,dx = (x^3-2)^{1/2} + C$

But how do we start with

$$\int \frac{3x^2}{2\sqrt{x^3-2}}\,dx \quad \text{or} \quad \int \frac{3}{2}x^3(x^3-2)^{-1/2}\,dx$$

and get the answer?

We notice the patterns and we use u-substition.

$$\int f'(g(x)) \cdot g'(x)\,dx = f(g(x)) + C$$

We'll substitute $u = g(x)$

and $du = g'(x)\,dx$

Caution: Do not ignore, overlook, or leave out the dx and/or du. You will always have an error, and sometimes it's catastrophic.

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**abstract structure**

$\int f'(g(x)) \cdot g'(x)\, dx$

**specific example**

$\int \underbrace{\frac{1}{2}(x^3-2)^{-1/2}}_{f'(g(x))} \cdot \underbrace{(3x^2\, dx)}_{g'(x)\,dx}$

| steps 1-2-3 Are about RECOGNITION |
|---|

↳ Once you're good at this, you'll do them in your head.

step 1: Notice two expressions multiplied together.
(Multiply may be disguised as divide.)
Constant coefficients may be aligned with either expression.

step 2: Notice that one of the two expressions is the derivative of the inside of the other expression.

$g(x)$ inside $\Rightarrow x^3-2$
$g'(x)$ outside $\Rightarrow 3x^2$

step 3: Identify the outside function $f'(x)$ — this is the function we will antidifferentiate in step 6.

$f'(x) = \frac{1}{2}(\text{○○○○○})^{-1/2}$

| Steps 4-5-6-7 Are written work |
|---|

step 4 a) Write u substitution: $u = g(x)$. $\quad u = x^3-2$
b) Differentiate the u-substitution and write in differential form: $du = g'(x)\, dx \quad du = 3x^2\, dx$.

Key: The du comes from your choice of u, not from the integral.

Check: The variables in du should be in the original integral. If they aren't, it's not correct yet.
[Missing coefficients we'll be able to fix.]

$\int \frac{1}{2}\underbrace{(x^3-2)}_{u!}^{-1/2} \cdot \underbrace{3x^2\, dx}_{du!}$ ☺ $3x^2$ was in the original.

step 5: Rewrite integral using u and du.
CAUTION: Every x should be replaced by u!
The problem should never have x and u at the same time.

$\int \frac{1}{2} u^{-1/2}\, du$

You may need to multiply by a constant inside and divide by that constant outside.

step 6: Anti-differentiate in u.

$= \frac{1}{2} \cdot \frac{2}{1} u^{1/2} + C$

$= u^{1/2} + C$

step 7: Substitute back, replacing u by g(x).

$= \boxed{(x^3-2)^{1/2} + C}$

Answer must be in the same variable as the original question.

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Integrate

(2) $\int (x^2-9)^3 \cdot 2x \, dx$

$u = x^2 - 9$ inside

$du = 2x \, dx$ $\frac{d}{dx}$ inside

$= \int u^3 \cdot du$ rewrite in u

$= \frac{1}{4} u^4 + C$ antidifferentiate

$= \boxed{\frac{1}{4}(x^2-9)^4 + C}$ substitute back

Same question with a slight alteration:

Integral property
$\int k \cdot f(x) \, dx = k \int f(x) \, dx$

(3) $\int x (x^2-9)^3 \, dx$

$u = x^2 - 9$ inside

$du = 2x \, dx$ $\frac{d}{dx}$ inside

$\frac{k}{k} = 1$

$\int \frac{k}{k} \cdot f(x) \, dx = \frac{1}{k} \int f(x) \cdot k \, dx$

We need $2x\,dx$ inside the integral in order to subst du.
But there is no 2!
We can multiply the inside by 2 so long as we multiply the outside by $\frac{1}{2}$ $\frac{1}{2} \cdot 2 = 1 \Rightarrow$ no real change.

$= \frac{1}{2} \int (x^2-9)^3 \cdot 2x \, dx$ mult inside by 2, outside by ½

$= \frac{1}{2} \int u^3 \, du$ rewrite in u.

$= \frac{1}{2} \left[ \frac{1}{4} u^4 \right] + C$ antidifferentiate.

$= \frac{1}{8} u^4 + C$ simplify

$= \boxed{\frac{1}{8}(x^2-9)^4 + C}$ Substitute back

(4) $\int u^2 (u^3+5)^4 \, du$

DANGER! We already have u in the problem!
Instead of using u-substitution, we'll pick some other letter and use it.

Let's do w-substitution.

CAUTION: Though it's legal to use x, most students forget to go back to u.

$w = u^3+5$

$dw = 3u^2 \, du$

$$= \frac{1}{3} \int \underbrace{(u^3+5)^4}_{w^4} \cdot \underbrace{3u^2 \, du}_{dw}$$

$\frac{1}{3} \cdot \frac{3}{1} = 1$. multiply by 1, so no change to question

$$= \frac{1}{3} \int w^4 \, dw$$

$$= \frac{1}{3} \cdot \frac{1}{5} w^5 + C$$

$$= \boxed{\frac{1}{15} (u^3+5)^5 + C}$$

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Integrate

⑤ $\int \sqrt[3]{(3-4x^2)} \cdot (-8x)\,dx$

$u = 3-4x^2$

$du = -8x\,dx$

$= \frac{-1}{8}\int -8\sqrt[3]{u}\,du$

$= \frac{-1}{8}\int u^{1/3}\,du$

$= \frac{-1}{8}\cdot\frac{3}{4}u^{4/3} + C$

$= \boxed{\frac{3}{32}(3-4x^2)^{4/3} + C}$

⑥ $\int \frac{x^3}{\sqrt{1+x^4}}\,dx$

$u = 1+x^4$

$du = 4x^3\,dx$

$= \frac{1}{4}\int \frac{4}{\sqrt{u}}\,du$

$= \frac{1}{4}\int u^{-1/2}\,du$

$= \frac{1}{4}\cdot 2u^{1/2} + C$

$= \boxed{\frac{1}{2}(1+x^4)^{1/2} + C}$

$= \boxed{\frac{1}{2}\sqrt{1+x^4} + C}$

⑦ $\int \frac{x^2}{(16-x^3)^2}\,dx$

$u = 16-x^3$

$du = -3x^2\,dx$

$= \frac{-1}{3}\int \frac{-3\cdot 1}{u^2}\,du$

$= \frac{-1}{3}\int u^{-2}\,du$

$= +\frac{1}{3}u^{-1} + C$

$= +\frac{1}{3}(16-x^3)^{-1} + C$

$= \boxed{\frac{+1}{3(16-x^3)} + C}$

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But what if the constants don't work out?
Remember the property of integrals:

$$\int k \cdot f(x)\,dx = k \cdot \int f(x)\,dx.$$

constant multiples can move in or out.
*CAUTION* No expression containing a variable can be moved.

Integrate.

(8) $\int x(5x^2+4)^3\,dx$

want $u = 5x^2+4$
$du = 10x\,dx$

but no 10 in question — so multiply by $1 = \frac{10}{10}$

$= \int \frac{10}{10} x(5x^2+4)^3\,dx$

$= \frac{1}{10}\int 10x\,(5x^2+4)^3\,dx$ move $\frac{1}{10}$ outside, leave needed 10 in.

$= \frac{1}{10}\int u^3\,du$

$= \frac{1}{10}\left[\frac{1}{4}u^4\right] + C$

$= \boxed{\frac{1}{40}(5x^2+4)^4 + C}$

(9) $\int t^3\sqrt{t^4+5}\,dt$

want $u = t^4+5$
$du = 4t^3\,dt$

$= \frac{1}{4}\int 4t^3\sqrt{t^4+5}\,dt$

$= \frac{1}{4}\int u^{1/2}\,du$

$= \frac{1}{4}\left[\frac{2}{3}u^{3/2}\right] + C$

$= \boxed{\frac{1}{6}(t^4+5)^{3/2} + C}$

(10) $\int \frac{x^3}{(1+x^4)^2}\,dx$

want $u = 1+x^4$
$du = 4x^3\,dx$

$= \frac{1}{4}\int \frac{4x^3}{(1+x^4)^2}\,dx$

$= \frac{1}{4}\int u^{-2}\,du$

$= \frac{1}{4}(-1)u^{-1} + C$

$= -\frac{1}{4}(1+x^4)^{-1} + C$

$= \boxed{\frac{-1}{4(x^4+1)} + C}$

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General Power Rule for Integration

$$\int [g(x)]^n \cdot g'(x)\,dx = \frac{[g(x)]^{n+1}}{n+1} + C \qquad n \neq -1$$

OR if $u = g(x)$

$$\int u^n\,du = \frac{1}{n+1} u^{n+1} + C \qquad n \neq -1$$

Do not get tripped up – NOT all problems in HW require substitution!

Integrate.

(11) $\int \left[x^2 + \frac{1}{(3x)^2}\right] dx$

$= \int \left(x^2 + \frac{1}{9}x^{-2}\right) dx$

$= \int x^2\,dx + \frac{1}{9}\int x^{-2}\,dx$

$= \frac{1}{3}x^3 + \frac{1}{9}(-1)x^{-1} + C$

$= \boxed{\frac{1}{3}x^3 - \frac{1}{9x} + C}$

* (13) $\int \sec^2(2x)\,dx$

$u = 2x$

$du = 2\,dx$

$= \frac{1}{2}\int \sec^2 2x \cdot 2dx$

$= \frac{1}{2}\int \sec^2 u\,du$

$= \frac{1}{2}\tan u + C$

$= \boxed{\frac{1}{2}\tan(2x) + C}$

(12) $\int 4\pi y\left(6 + y^{3/2}\right) dy$

$= 4\pi \int \left(6y + y^{5/2}\right) dy$

$= 24\pi \int y\,dy + 4\pi \int y^{5/2}\,dy$

$= \frac{24\pi}{2}y^2 + 4\pi \cdot \frac{2}{7}y^{7/2} + C$

$= 12\pi y^2 + \frac{8\pi}{7}y^{7/2} + C$

$= \boxed{\frac{4\pi}{7}y^2\left(21 + 2y^{3/2}\right) + C}$

(14) $\int x \sin x^2\,dx$

$u = x^2$

$du = 2x\,dx$

$= \frac{1}{2}\int \sin(x^2) \cdot 2x\,dx$

$= \frac{1}{2}\int \sin u\,du$

$= -\frac{1}{2}\cos u + C$

$= \boxed{-\frac{1}{2}\cos(x^2) + C}$

(15) $\displaystyle\int \frac{\sin x}{\cos^3 x}\,dx = \int \frac{\sin x}{[\cos x]^3}\,dx$

*

$u = \cos x$

$du = -\sin x\,dx$

$\displaystyle = -\int \frac{-\sin x\,dx}{\cos^3 x}$

$\displaystyle = -\int \frac{1}{u^3}\,du$

$\displaystyle = -\int u^{-3}\,du$

$= -\left(\frac{-1}{2}\right)u^{-2} + c$

$= \frac{1}{2}(\cos x)^{-2} + C$

$= \boxed{\dfrac{1}{2\cos^2 x} + c}$

(16) Solve the DE.

$f'(x) = 2x(4x^2-10)^2$

$f(2) = 10$

$\displaystyle f(x) = \int 2x(4x^2-10)^2\,dx$

$u = 4x^2 - 10$

$du = 8x\,dx$

$\displaystyle = \frac{1}{4}\int (4x^2-10)^2 \cdot 8x\,dx$

$\displaystyle = \frac{1}{4}\int u^2\,du$

$= \frac{1}{4}\cdot\frac{1}{3}u^3 + c$

$f(x) = \frac{1}{12}(4x^2-10)^3 + c$

$f(2) = \frac{1}{12}(4\cdot 2^2 - 10)^3 + c = 10$

$\frac{1}{12}(6)^3 + c = 10$

$c = 10 - \frac{1}{12}(216)$

$c = 10 - 18$

$c = -8$

$\boxed{f(x) = \frac{1}{12}(4x^2-10)^3 - 8}$

(17) Integrate

$\displaystyle\int \frac{\sec^2 x}{\tan^3 x}\,dx$

$u = \tan x$

$du = \sec^2 x\,dx$

$\displaystyle = \int \frac{1}{u^3}\,du$

$\displaystyle = \int u^{-3}\,du$

$= \frac{-1}{2}u^{-2} + c$

$= -\frac{1}{2}(\tan x)^{-2} + c$

$= \boxed{\dfrac{-1}{2\tan^2 x} + c}$